**T.C.**

**SÜLEYMAN DEMİREL UNIVERSITY**

**GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES**

**SOLVING CONNECTION PROBLEMS IN MOUNTAINS BY COOPERATIVE GAME THEORY**

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**THE DEGREE OF MASTER OF SCIENCE**

**DEPARTMENT OF MATHEMATICS**

**ISPARTA - 2016**

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**APPROVAL OF THE THESIS**

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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# ABSTRACT

**M.Sc. Thesis**

**SOLVING CONNECTION PROBLEMS IN MOUNTAINS BY COOPERATIVE GAME THEORY**

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This thesis consists of five chapters. In the first chapter, a summary of game theory, historical overview and information of the Shapley value is given. Also connection situations related to game theory, the aim of the thesis is studied. In the second chapter, we explain the connection problems in mountains and study how to find the minimum cost-spanning tree for directed weight of graph. In chapter 3, applications in cooperative games are given and the Shapley value is calculated. In the fourth chapter, we link the main areas in the city of Isparta with the Ataturk stadium. In the last chapter, the conclusion of the study and some suggestions are given.

**Keywords:** Cooperative games, mountains situations, connection problems, solution concepts, Shapley value.

**2016, 59 pages**

# ÖZET

**Yüksek Lisans Tezi**

**DAĞLARDAKİ BAĞLANTI PROBLEMLERİNİN İŞBİRLİKÇİ OYUN TEORİSİ İLE ÇÖZÜMÜ**

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Bu tez beş kısımdan oluşmaktadır. İlk kısımda oyun teorisi ile ilgili özet, tarihsel bakış ve Shapley değeri ile ilgili bilgi verilmiştir. Ayrıca, bağlantı durumundan çalışılmıştır. İkinci bölümde dağlardaki bağlantı durumları açıklanmış ve ağırlıklı bir çizgeden minimum bağlantılı bir ağaç oluşturulması çalışılmıştır.

Üçüncü kısımda işbirlikçi oyunlara uygulamalardan bahsedilmiş ve Shapley değeri hesaplanmıştır. Dördüncü kısımda, Atatürk stadyumuna Isparta'nın belli bölgelerinin bağlanması gösterilmiştir.

Son kısımda, çalışmanın sonuçları ve bazı öneriler verilmiştir.

**Anahtar Kelimeler:** İşbirlikçi oyunlar, çözüm yöntemleri, Shapley değeri, dağ durumları, bağlantı problemleri.

**2016, 59 sayfa**

# ACKNOWLEDGMENTS

I want to put this page lively moments of the conclusion of this study, and I am proud of them and seize the opportunity to thank those people who were necessary to do this work.

I would like to express my deep gratitude to my supervisor, Assoc. Prof. Dr**.** Sırma Zeynep ALPARSLAN GÖKfor her guidance and encouragement throughout the process of writingthe thesis. Her invaluable comments on the chapters of the thesis enrich the whole work and for her words that always encourage me to work hard. I feel indebted to Dr. Osman Palancı for his helpful and his brotherly treatment, and for his invaluable knowledge. This thesis is a successful result of the joint work with my supervisor Assoc. Prof. Dr. Sırma Zeynep ALPARSLAN GÖKand Dr. Osman PALANCI all my sincere gratitude and appreciation to them again. I would like to express my profound gratitude to the head of the department of mathematics and all professors of the department.

I dedicate this work to the spirit of my late father (Farhan Ashore), who was the reason to succeed for all the family and after that I am greatly indebted to my family, especially my mother who kept encouraging me to do my best to get my (M.Sc.) degree. I would like to thank my brothers: Ahmed, Mahmoud, Mohammed, Ali, Husain, Hessen, Abbas, Ehsan, and Wessam, my sisters: Badria, Sabria, Shaima, thanks to my wife and my sons: Saba, Ahmed, and Ibo-AL noon (Mahmoud). Thanks to all my friends who have supported me and advised me.

Hamid ALZAKI

ISPARTA, 2016

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# 

# LIST OF SYMBOLS AND ABBREVIATION

n-dimensional real vector space

The set of complex numbers

The set of all real numbers

The set of all natural numbers

The set of rational numbers

The set of integer number

Subset: A has some (or all) elements of B

Proper Subset: A has some elements of B

Not a Subset: A is not a subset of B

Superset: A has same elements as B, or more

Proper Superset: A has B's elements and more

Not a Superset: A is not a superset of B

Complement: elements not in A Union: in A or B (or both) Intersection: in both A and

Difference: in A but not in B Equality: both sets have the same members

Cartesian Product: set of ordered pairs from A and B

Cardinality: the number of elements of set A

Element of: a is in A Not element of: b is not in A

The empty set

Smaller or equal to

Greater or equal to

The set of players

A player

The game

The set of all cooperative games

Cooperative game

Characteristic function

The value of the coalition S

The set of subset of N

The set of all permutations (possible orderings) of players *N*

A permutation

Marginal contribution vector

Shapley valaue

Any permutation

A rooted directed weight graph

Link between *i* and *j*

The heigh of the vertix *i*

Minimum cost spanning tree

Sum

**A** Matrix of a system (adjacency matrix)

# 1. INTRODUCTION

## 1.1. Motivation

In general, any problem is phenomenon which needs to interpret and to solve. No research can achieve its own goal without a problem to discuss and a suitable solution for it. Since ancient times inventions and discoveries occur as a result of the need for people to continue their lives in an acceptable manner. For example people’s fear from monsters made them think to built houses to protect themselves also their needs to heat made them look for generating fire.

The researchers choose this topic after earthquake happened in Isparta city in the south west of Turkey where they are living. On the other hand, the state of fear and panic hit people in the mountain regions. This problem motivated us to use cooperative game theory to try solve this problem in order to help people to access to a safe place in a shortest way and with minimum cost.

Connection cases take place from existence of a group of agents, each of which needs to be connected to a source. If links are costly, then agents evaluate the chance of cooperating for reduce costs. In such cases more studies on the problems of linking for different purposes are done to get optimal solutions. The cost allocation problem arising from a connection situation is introduced by Claus and Kleitman Claus (1973) and has been studied with assistance of cooperative game theory since the paper of Bird (1976) We note that cost allocation problems may arise on many different physical networks such as telephone lines, highways, electric power systems, also Moretti et al., (2002) They conducted a Study on the connection problems in mountains and model of cooperative cost game (Alparslan Göket al., 2014).

In this study let's suppose a group of people whose houses located in mountain areas, suffering from the earthquake, the houses are connected with many roads leading to city center and can be evacuated to a safe place with minimum cost and shortest way. We use cooperative game theory and the Shapley value to solve this problem.

Now let's talk about game theory. An interesting part of applied mathematics is the game theory. Game theory is an important exercise for our everyday life. Game theory A mathematical method of decision-making in which a competitive situation is analyzed to determine the optimal course of action for an interested party, often used in political, economic, and military planning. Also called theory of games. By this it is assumed that all participants behave rationally. Our works or business for some cases, depends not only on what we do, but also on what others are doing. Some examples of games in game theory are the negotiation of a price with a seller, the vote at a presidential election, the participation in an auction on the internet and even for trying to find a seat on a bus. Other most known games are chess, football, monopoly, etc.

Game consists of two players or multi-players who have conflicting or common interests. All the players can have a strategy in order to move for a situation. Game theory text on, mathematical operation for choosing an optimal response to player facing the rival player. Game theory can be defined as a set of tools to be developed to analyze the interactions among number of agents to reach their goals (Nikolidaki and Dimitris, 2015).

Game theory has been applied to many fields such as biology, economics political science, law, sociology, and computer science . In this thesis, game theory is central for studying the interaction among decision makers (which are called players) in connection situations, where players need to be connected directly or via other players to a source, and where connections between players and the root are costly. Since the seminal book of Theory of Games and Economic Behavior, it is usual to divide Game Theory into two main groups of interaction situations (which are called games), non-cooperative and cooperative games. Non-cooperative games deal with conflict situations where players cannot make binding agreements. In cooperative games all kinds of agreement among the players are possible. In this thesis we discuss and analyze solving connection problems in mountains by using cooperative game theory (Fragnelli, 2008).

## 1.2. History Point of Game Theory

Game theory is a branch of applied mathematics developed through ancient and modern studies, it has been able to attract many of the world’s biggest mathematicians, and won in two Nobel Prizes and is even credited with winning the Cold War. The origins of game theory go far back in time. Recent work suggests that the division of an inheritance described in the Talmud (in the early years of the first millennium). In the early time of the seventeenth century, such professional mathematicians as Christiaan Huygens (1629-1695) and Gottfried W. Leibniz (1646-1716) suggested the inventation of a new field that would make use of the scientific method to study human conflict and interactions. During the nineteenth century, several leading economists created simple mathematical examples to analyze particular examples of competitive situations. The first general mathematical theorem in this subject was proved by the distinguished mathematician Ernst Zermelo (1871-1956) in 1912. It stated that any finite game with perfect information, such as tic-tac-toe, checkers chess, has an optimal solution in pure strategies, which means that no randomization or secrecy is necessary A game is said to have exact information if at each stage of the play, every player is awake of all past moves by himself and others as well as all future choices that are allowed. This theorem is an example of an existence theorem which demonstrates that there must exist a best way to play such a game, but it does not provide its construction that is detailed plan for playing (http://www.math.uri.edu/~kulenm/mth381pr/GAMETH/gametheory.html). Timeline of the history of the game theory as shown in Figure 1.1.

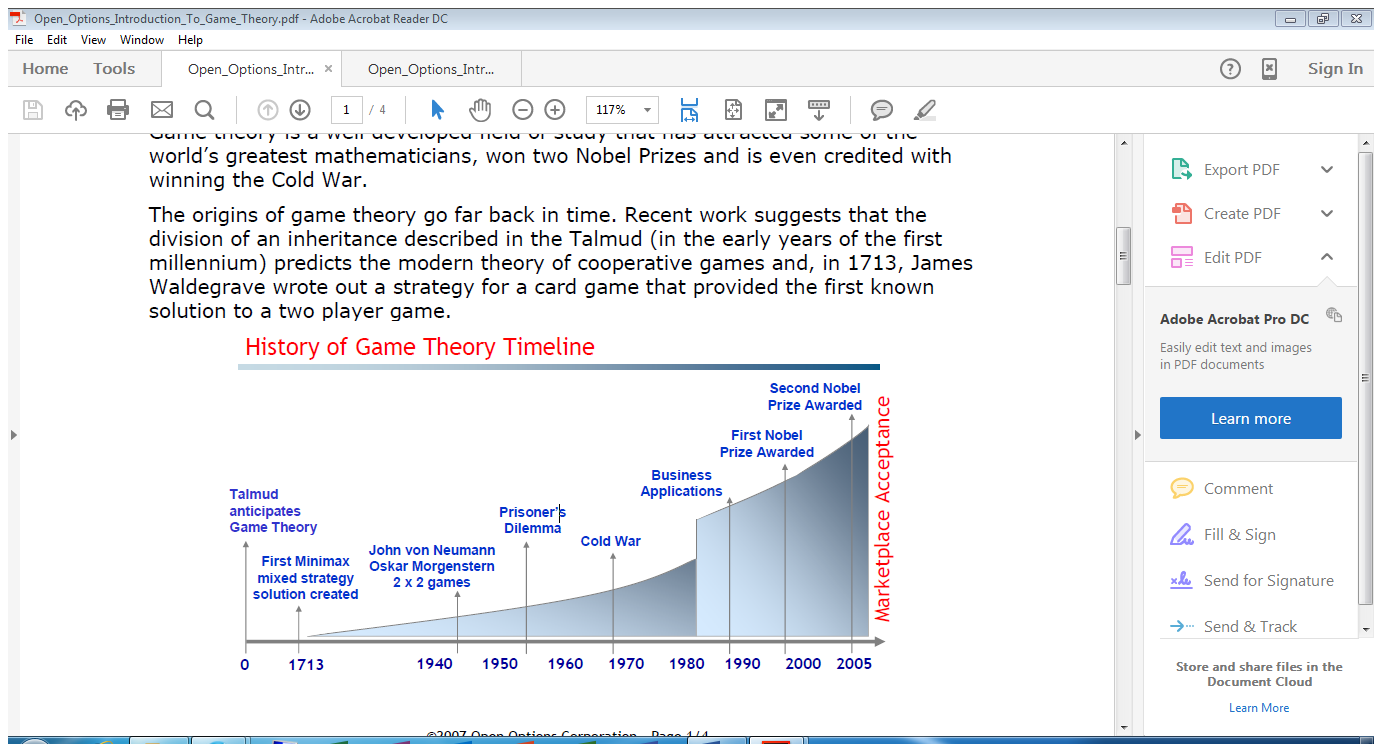


Figure 1.1. History of game theory timeline

The mathematician Emile Borel (1871-1956) introduced the notion of a mixed, or randomized, strategy when he investigated some elementary duels around 1920. John von Neumann proved in 1928, that every two-person, zero-sum game must have optimal mixed strategies and an expected value for the game. His result was extended to the existence of equilibrium outcomes in mixed strategies for multiperson games that are either constant-sum or variable-sum by John F. Nash Jr. (1931-), in 1951. Modern game theory dates from the famous minimax theorem that was proven by von Neumann. He gradually expanded his work in game theory, and with co-author Oskar Morgenstern. They wrote the classic text Theory of Games and Economic Behaviour (1944) and introduced the first general model and solution concept for multiperson cooperative games, which are primarily concerned with coalition formation (economic, voting blocs, and military alliances) and the resulting distribution of gains or losses. The Nobel Memorial Prize in Economics is awarded to three game theorists in 1994. The recipients were John C. Harsanyi (1920- ) of the University of California, Berkeley, John F. Nash of Princeton University, and Reinhard Selten (1930-) of the University of Bonn, Germany. Game theory has provided important theoretical foundations in economics, political science especially in the study of voting, elections, and international relations. In addition, game theory has contributed major insights in environmental sciences, particularly in understanding the evolution of species and conditions under which animals fight each other for territory or act cooperatively. Conflict has been a major thread throughout human history. It arises when two or more person, with different values, compete to try to control the course of events. Game theory uses mathematical sciences to study cases that contain cooperation and conflict (www. Rijpm. Com/.../Open\_ Options\_Introduction\_To\_Game\_Theor).

The famous problem in non cooperate game theory is Prisoners' dilemma. There are two prisoners whose aim is to minimize the years of imprisonment. They have committed a crime jointly. Each prisoner is interviewed separately and there are not any contacts whatsoever between them. They decide individually to confess or not confess the crime taking into account possible decisions of the other prisoner (strategic game). Each prisoner chooses his dominant strategy, that is the behaviour giving the best result regardless of the decision of the other prisone. For example:

Lets say Mr. A and Mr. B has been arrested for some minor crime, the police think they committed another minor crime but they do not have enough evidence to convict them. After that, the police detain them in separate rooms so they can not talk, and play a little game. Regardless of what the other decides, each prisoner gets a higher reward by betraying the other. There are two possibilities when interrogated them either confess or not confess (if both of them confess, they prison 3 years), (if booth of them not confess, they will prison 2 years), ( if one of them confessed and the other does not confess then who confesses the imprisone one year who does not confess jails four years) see Table 1.1.

Table 1.1. Years of prison for A and B

|  |  |  |  |
| --- | --- | --- | --- |
| Prisoner B | | | |
|  | | Confess | | | Not confess | |
| Prisoner A | Confess | | 3 | 3 | 1 | 4 |
| Not confess | | 4 | 1 | 2 | 2 |

The first number shows the imprisonment years of A, the second number of B. For example if A confesses and B deny, A gets 1 year imprisonment and B gets 4 years (field at the top right).

* Players: decision makers (prisoner A or B).
* Strategy: behaviour of the players (confess ornot confess).
* Payoff: result (years prison ).
* Dominant strategy: the best outcome for a player regardless of the decision of the other player.
* Which are the dominant strategies in this game?

From the point of view of prisoner A.

* If B confesses, I should also confess (3 years are less than 4 years).
* If B did not confess, I should again confess (1 year is less than 2 years).
* strategy of A: I confess whatever decision of B. Confess is the dominant strategy of A (3 years imprisonment).

From the point of view of prisoner B.

* If A confess, I should also confess (3 years are less than 4 years).
* If A denies, I should again confess (1 year is less than 2 years).
* Strategy of B: Confess is his dominant strategy, too (3 years prison)
* Therefore, not confess for A and not confess for B is the Nash Equilibrium is the best solution.

If both prisoners could cooperate successfully, they would get a better outcome for both (2 years imprisonment). But they cannot cooperate, thus, the dominant strategy is the best result which can be achieved when deciding individually. That is the dilemma of the prisoners: By cooperation they could get a better result than by deciding individually.

The prisoners' dilemma characterizes many economic decisions where only a few participants have to decide individually and where the outcome is influenced not only by the own decision but also by the decisions of the other participants. This is often in oligopolistic situations the case. Cooperative game is the main subject of the thesis we will clarify it later in this chapter (Neyman, 1985).

## 1.3. Game Theory and Connection Situations

Game Theory is a key factor for studying the dealing among decision makers (the players) who need to find the best solution and less cost to connect direct and indirect regions with the source according to the geographical nature of the city under consideration. It is usual to divide Game Theory into two main groups of interaction situations (which are called games), non-cooperative and cooperative games. Non-cooperative games deal with conflict situations, where players can not make binding agreements. In cooperative games all kinds of agreement among the players are possible. In non-cooperative games, each player choose to act in his own interest keeping into account that the outcome of the game depends on the actions of all the players involved. Actions can be made simultaneously by players, as in the stone, paper, scissors game or in matching pennies, or sequentially at several time moments, as in chess. Cooperative games deal with situations where groups of players (which are called coalitions) coordinate their actions with the objective to end up in joint payoffs which often exceed the sum of individual payoffs. There are two fundamental questions that cooperative game theory considers:

Who will cooperate with whom that is, which coalitions will form?

After coalitions have formed and earned the value defined by the characteristic function, how will they divide this value among themselves? Also how will they divide the value fairly?

It is explained in detail with the examples as well as the solution of the problems to this study in Chapter 4. Analysis of a simple example of the problem of the distribution of the costs arising from the connection cases is given below. For example, linking between two neighboring towns such as DIYALA (Baquba) and SALAHADDIN (Tikrit) in Iraq may help to implement a joint refuse collection system. Diyala could implement a system for itself at a cost of 7 million dollars, while Tikrit could implement its refuse collection system at a cost of 4 million dollers. However, if the two cities cooprate, they can achieve this system in 10 millon dollars (Suijs 1998). This case can be constructed as a cooperative cost game such that } is the set of players. Here, the first player is Diyala and the second player is Tikrit. The characteristic cost function assigns to each coalition the corresponding cost of implementing a refuse collection system, (in million dollars) as described in Table 1.2.

Table1.2. Cost of implementing a refuse collection system.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| S |  | {1} | {2} | {1,2} |
|  | 0 | 7 | 4 | 10 |

The map of Iraq where the two cities are cooperating is shown in Figure 1.2.



Figure 1.2. Map of Iraq

Clearly, it makes sense to cooperate since the two players can jointly save 1 million. Cooperation can only take place, however, if they agree on how to share the total cost of 10 million dollars. Trying to solve this problem, a cost allocation that can be accepted by both towns 1 and 2 must be effcient (the total cost must be entirely shared), equitable. On the other hand, town 1 may not accept to the proportional allocation method to waste production, because 8 million more than the cost of applying its own system. One possible solution for cost game is to equally divide the amount of money that 1 and 2 saved by the cooperation. Using this method, town 1 would pay million, and town 2 would pay million, and this provide motive to cooperation. For example, one could propose to share equally the cost of 10 million dollar, 5 million dollar for each city. This case for equal division is that each city has an equal quantities of waste to enter in an agreement, so each town should support an equal load. On the other hand, it could be the case that town 1 produces four times the waste of town 2. Then, it seems fair to propose a method based on the proportion of waste produced by the two towns. In this allocation method, town 1 pays 8 million dollar and town 2 pays 2 million dollar. In fact, town 2 is not likely to agree to equal division.This allocation gives to players an incentive to cooperate. Any allocation in which player 1 pays at most 7 million and player 2 pays at most 4 million creates no disincentives to cooperation. By using game theory terminology, such an allocation is stable. The set of all stable allocations is the core of the cost game (Moretti, 2009).

## 1.4. Cooperative Game Theory

Game theory is the study of decision problems with multiple decision makers. It is divided into two sections, non-cooperative game theory and cooperative game theory While the non-cooperative game theory analyzes the situation where all the action is taken by the agent and one independent, cooperative game theory studying mainly interactions among coalitions of agents. The main question of the theory of cooperative game is the following: because of to a number of possible outcomes for each coalition, what result will happen? The answer is accurate, through solutions for a number of applications and analysis them. From each of these points of view, we discuss the most widely used solution concepts: the core and the Shapley value. The core and the Shapley value concepts Have proved successful each in its own way. The literature on the core proposes a foundation for Walrasian equilibrium of large economies. First, the core of replicated economies converges to the set of Walrasian allocations (Debreu and Scarf 1963). Second, there are meaningful sufficient conditions that guarantee the non-emptiness of the core (Bonda Reva (1963), Shapley (1967), Scarf (1967)). The Shapley value, which is first proposed in Shapley (1953), has been used not only to recommend reasonable rules of allocation, but also used as a practical tool to measure voter power and to allocate cost. We give some definitions and results concerning classical cooperative game theory needed in the thesis. For an extensive description of classical game cooperative theory see Shapley (1953), Curiel (1997), Tijs (2003) and Branzei vd. (2008). Cooperative game theory is concerned primarily with coalitions groups of players who coordinate their actions and pool their winnings.

**Definition 1.4.1. (A** **cooperative game)** in characteristic function form is an ordered pair consisting of the player set and the characteristic function with (Dubey, 1975).

Elements of the set N are called players and the relevant set-function the characteristic function of the game. A subset S of the player set N (notation: ) is called a coalition and the worth of coalition S in the game. The player set N itself is also called the grand coalition, whereas a coalition S is said to be nontrivial if The number of players in a coalition S is denoted by. Generally, we shall identify the cooperative game with its characteristic function . Further, the class of all cooperative n-person games with player set is denoted by . Cooperative game in characteristic function form is usually referred to as a transferable utility game (TU-game).

**Example 1.4.2. (Glove Game)** Let consisting of two disjoint subsets and The members of L possess each one left-hand glove, the members of R one right-hand glove. A single glove is worth nothing, a right-left pair of gloves is worth 10 Turkish Liras. This situation can be modeled as a three-person game with and otherwise.

The family of cooperative game with player set N forms with the usual operations of addition and scalar multiplication of functions )dimensional lineer space. A basis of this space is supplied by the unanimity games.

**Definition 1.4.3.** For each **(the unanimity game )** is defined by

The set is a basis of the linear space . One can easily check that for each we have

Here, the coefficients of are also called Harsanyi dividends (Harsanyi, 1959). So, every cooperative game can be written in terms of . For cost games the basis is and

(dual S-unanimity game)

Let be the set of all perrmutations of N. The set consists of all predecessors of with respect to the permutation

**Definition 1.4.4. (Marginal Contribution)** Let The marginal contribution with respect to has the coordinate the value

**Definition 1.4.5. (Monotonic Game)** A game is said to be monotonic if

**Definition 1.4.6. (Nonnegative Game)** A game *is* said to be non-negative if for each we have .

**Definition 1.4.7. (Additive Game)** A game is additive if for all

**Definition 1.4.8. (Convex Game)** A game is convex if for each

**Definition 1.4.9. (Concave Game)** A game is concave if for each .

**Definition 1.4.10. (Superadditive Game)** A game is superadditive if for each

In a superadditive game, it is advantageous for the players to cooperate. Most of the cooperative games arising from real life situations are superadditive games (Alparslan Gök et al, 2009). Indeed it may be argued that if forms, its members can decide to act as if S and had formed separately.

Now, we turn to one of the basic question in the theory of cooperative TU-games if the grand coalition forms, how to divide the profit or cost savings ? This question is approached with the aid of solution concepts in cooperative game theory like stable sets (Von Neumann and Morgenstern, 1944), the Shapley value (Shapley, 1953), cores (Gillies, 1959) the nucleolus (Schmeidler, 1969) the T-value (Tijs, 1981). Hence, a solution concept assigns to a coalitional game at least one payoff vector

where is the payoff allocated to player

We mention now some desirable properties for one-point solution concepts. Extensions of these properties to set-valued solution concepts are straightforward.

Suppose the grand coalition N forms, and they then obtain the value *v*(N). The next question to be answered is how the value should be divided among the players . The Shapley value provides a principled way to do this. It proposes that each player should be given an amount that satisfies the following axioms :

Let Then satisfies

**Axiom 1.4.11**. Individual rationality if for all and N.

**Axiom 1.4.12**. Efficient if (v)= *v* (N) for all *v* ∈.

**Axiom 1.4.13**. Dummy player property if for all and for all dummy players in , .e. players N such that for all .

**Axiom 1.4.14.** Additivity if for all .

Formally, player i contribution to a coalition is simply or the amount extra that could obtain if they admitted player as a member. If this value is 0, then there is no benefit to be obtained. Given this definition, the symmetry axiom, for example, means that two players should receive the same value if they make the same contribution to all coalitions.

These axioms, however, say nothing about how to compute such a value (or even whether there is any value that satisfies them). In a remarkable result, Shapley showed that there is a unique solution to these axioms. The basic idea is this: imagine all the possible orders in which the grand coalition could form, one player at a time.

If then the grand coalition could form in six possible ways. or

Then, Shapley suggested, a player should receive the average contribution that he or she makes, over each of these orderings, to the set of players that precedes him or her in the ordering. If for example, then is , where

Now, if we compute the Shapley values in this way, the resulting payoff vector () satisfies the four axioms; and what is more, it’s the only payoff vector that satisfies them average over 3! = 6 orderings; and even in this trivial case the arithmetic starts to become boring. So, computing the Shapley value by directly averaging a player’s contribution over all orderings of the players is not going to be practicable in general. (Chalkiadakis, 2011).

One of the most important solution concepts in cooperative game theory is the Shapley value. The Shapley value assocaties to each game one payoff vector in

**Definition 1.4.15. (Shapley value)** The Shapley value of a game is the average of the marginal vectors of the game, i. e.

(1.4.1)

With the aid of (1.4.1) one can provide a probabilistic interpretation of the Shapley value as follows. Suppose we draw from an urn, containing the elements of a permutation (with probability ). Then we let the players enter a room one by one in the order and give each player the marginal contribution created by him. Then, for each , the -th coordinate is the expected payoff player according to this random procedure. One can rewrite (1.4.1) as follows:

**Example 1.4.16.** Let and the characteristic functions be .

Now, we compute the marginal vectors of this game. For this, we can check the permutations of N. The permutations of the elements of N is

When

Then the marginal vector for is

When

Then the marginal vector for

When

Then the marginal vector for is (7,0,1).

When

Then the marginal vector for is

When

Then the marginal vector for is

When

Then the marginal vector for is

We illustrate the results in the following Table 1.3.

Table 1.3. The marginal vector of the Example 1.4.16

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| (1,2,3) | 0 | 7 | 1 |
| (1,3,2) | 0 | 4 | 4 |
| (2,1,3) | 7 | 0 | 1 |
| (2,3,1) | 5 | 0 | 3 |
| (3,1,2) | 4 | 4 | 0 |
| (3,2,1) | 5 | 3 | 0 |

## 1.5. The Aim of The Thesis

The idea behind game theory is to interpret a payoff for each game. Non-cooperative game theory has seen many interesting results and applications. The aim of this paper is to introduce a simple study for cooperative game theory. We can find payoffs or costs by computing Shapley value. We can show the usefulness of our approach by way of simple examples such as the glove game. Also, we find some general results. With regard to this study about solving connectivity problems in the mountains, the aim is reducing the cost to citizens in normal cases, use game theory to solve the transportion problems in emeregency cases, identified the suitable method to link the houses in the mountains regions in order to preserve the environment, how to use applied mathematics in public life and this study can be generalized in many places of the world and equitable distribution of costs.

# 2. CONNECTION PROBLEMS IN MOUNTAINS

## 2.1. Introduction

Sometimes mountainous areas need services different from the rest of the other areas. The problems that occur in these areas and ways to solve them encourages researchers and scholars in applied mathematics field. To study such problems and reaching the optimal solution in serve the community and contribute to the economic activities. In this study, we will use cooperative game theory to solve connection problems in mountains. Many of researchers have studied and discussed such problems as where they consider a group of people whose houses lie on mountains which surround a valley or a part of a coast. Their houses are not yet connected to a drainage where one has to empty their sewage. Obviously sewage has to be collected downhill in a water purifier in the valley or along the coast, where it has to be purified before introduction into the environment (Moretti et al., 2002).

One solution for the houses is to get rid of the waste water immediately, so each one wants to connect his house with a drain pipe to the water purifier. However, it is possible but not necessary for everyone to be connected directly with the water purifier, being connected via others is sufficient. Assuming that pipes are large enough one pipe can serve more than one person (see Figure 2.1.). The network drawn in the picture is a directed weighted graph, whose vertices are the houses, whose root is the water purifier and whose edges are the drain pipes which are allowed to be built. They solve this problem by finding a spanning tree with minimum costs and model of cooperative cost game.

On the other hand, employing pumps to send sewage from houses at lower heights to houses at upper heights could be too expensive. Also, practical reasons due to the inhomogeneous consistency of the waste water could suggest not to employ pumps. Therefore, exploiting gravity, only connections from houses to strictly lower ones are allowed (connections between houses at the same height are not allowed in order to avoid dangerous stagnation). A possible situation is sketched in Figure 2.1. The network is drawn.



Figure 2.1. A possible mountain situation

Figure 2.1. is a directed weighted graph, whose vertices are the houses, whose root is the water purifier and whose edges are the drain pipes which are allowed to be built. The numbers indicate the cost of building the corresponding pipe. Sometimes a connection from higher houses to lower houses is impossible (e.g. because of a natural reef between the two houses), as for example the connection from house 3 to house 2 in Figure 2.1. However, it is always possible to connect a house directly with the root.

A mountain situation as described above leads to a connection problem of a directed graph without cycles and with some other properties. In the next section we consider such connection problems and describe a simple method to find a spanning tree with minimum costs.

## 2.2. Minimum Cost Spanning Tree Situations And [New Terminology](http://cpsc.ualr.edu/srini/DM/chapters/review5.3.html#New%20Terminology%28graph%20terms%29)

The goal in minimum cost spanning tree problems is to build of a network with minimal costs which states for every vertex in the network which has connection with the source. Examples of minimum cost spanning tree problems are the problem of building a network of computers that connects every computer with some server or the problem of building a drainage system that connects every house in a city with the water purifier. Another example of a minimum cost spanning tree problem is the problem of the using of cars. Suppose that three employees consider the possibility of sharing the cost of transport in order to reduce daily travel costs. The cost of driving a car from one employee to another or from one employee to the company are given in Figure 4. Here, the employees are denoted by 1, 2, and 3 and the company by 0. A minimum cost spanning tree in this network is the set of edges {01, 12, 13} with cost 18. This tree corresponds to the plan of carpooling in which employees 2 and 3 drive their car in solitude to employee 1 where all employees take one car in order to drive together to the company (Norde et al., 2001).



Figure2.2. The cost of driving and a minimum spanning tree

Having solved the problem of finding a minimum cost-spanning tree the employees are confronted the problem of how to divide the cost of 18 among the employees in a fair way. At this stage, cooperative game theory enters the scene. The employees consider the minimum cost-spanning tree game where

and is the characteristic function which computes for every the cost of a network of minimal cost connecting every employee in S with the company. So,and, e.g. since, the set of edges is a minimum spanning tree for . First one has to find a minimum spanning tree, which forms in every step of the algorithm an edge between a node which is not connected yet with the source and the source or a node which is already connected with the source. Secondly, a solution assigns the cost of an edge which forms in some step of the algorithm to the node which gets a connection with the source in that same step. In Figure 4 the algorithm first forms edge , then edge and finally edge and an allocation

**Definition 2.2.1. (An undirected graph)** is a pair where is a set of vertices or nodes and is a set of edges e of the form with The complete graph on a set of vertices is the graph where ( Moretti et al., 2004).

A path between i and j in a graph is a sequence of nodes (), where and such tha t for each s and such that all these edges are distinct. A cycle in is a path from i to i for some A path is without cycles if there do not exist such that . Two nodes are connected in if or if there exists a path between and in E. A connected component of in is a maximal subset of with the property that any two nodes in this subset are connected in

This thesis deals with minimum cost spanning tree (mcst) situations, i.e. situations, where a set of agents is willing to be connected as cheap as possible to a source (i.e. a supplier of a service) denoted by 0, based on a given weight (or cost) system of connection. In the sequel, we use also the notation and for the weight function, i.e. a map which assigns to each edge a non-negative numberrepresenting the weight or cost of edge e.

**Definition 2.2.2.** **(A directed graph)** allows connection between vertices only in the direction indicated by the arrowed path lines, which are called directed edges.

**Definition 2.2.3.** **(A weighted graph)** is a direct graph which has values on the directed edges (Gansner, 1993).

These values, for exampele represent the cost of travelling from one vertex to the next. The cost can be measured in many terms, depending upon the application. A graph of airport stations and flight lines between those stations can be seen in Figure 5(a), (b) and (c). The only difference is the meaning of the weights of the edges. Graph (a) represents the flight distance between stations. The graph (b) represents the average flight time in minutes, and the graph (c) represents the dollars costs of a plane ticket between the areas. Looking at graphs (a) and (b), one can see the relative cost of a direct flight from area 1 to area 3 is cheaper in terms of time and distance compared with the indirect route of travel from areas 1 to 2 to 3. However, in graph (c), the ticket costs of an indirect way is cheaper than the direct flight.

125

110

125

100

200

150

Figure 2.3. (a). The distance in miles

45

30

35

30

65

55

Figure 2.3. (b). Travel time in minutes

100

45

75

50

200

85

Figure 2.3. (c). Ticket cost

**Definition 2.2.4.** **(A weighted adjacency matrix)** is a matrix representation of a weighted graph (Umeyama, 1988).

Let and be numbered vertices for Let be the weight of a directed edge, if there exist an edge and assume that all edges weights are positive, . Suppose be an adjacency matrix, where , if there exist an edge .

For each vertex *i* in the Figure, , and where no path exists, An adjacency Matrix for (Figure 5. (c)) is show in Table 2.1.

100

50

75

45

200

85

Figure 2.3. (a,b,c). Weight digraph

Table 2.1. Adjacency matrix for Figure 2.3. (c).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| A | 1 | 2 | 3 | 4 |
| 1 | 0 | 100 | 200 | 50 |
| 2 |  | 0 | 75 |  |
| 3 |  |  | 0 |  |
| 4 |  | 45 | 85 | 0 |

## 

## 2.3. Warshall's Algorithm

Warshall's algorithm finds the shortest path between every pair of vertices in digraph using Boolean operators and matrices. Begin by creating an adjacency matrix **A** for Figure 6. As before, with one notable difference instead of using weights (and you will have noticed by now that none are assigned to digraph in figure below), we use Boolean operators. That is to say, if there is a path, enter a 1 in matrix A, and enter 0 if no path exists (Lu, Hongjun et al., 1987), ‏ as shown in Table 2.2.

2

1

1

3

4

4

5

Figure 2.4. Warshall's algorithm

Table 2.2. Matrix A

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A | 1 | 2 | 3 | 4 | 5 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 0 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 |
| 5 | 1 | 0 | 0 | 0 | 0 |

This matrix tells us whether or not there is a path p of length 1 between two adjacent vertices. Building upon matrix A, we creates a new matrix for which we choose one vertex to act as a pivot be an intermediate point between 2 other vertices. Initially, we choose vertex 1 as our pivot for . The value we are seeking is that of For vertices and  is one of the following. (1, if there exists an edge between vertices and , or if there is a path of length from to and from to else) (0, if there is no path).

First, realize that all paths of length 1 between vertices and are already established. We are searching now for any paths of length 2 which use vertex 1 as a pivot point. The only way vertex 1 can be a pivot is if a path already lies between some vertex and 1, and between 1 and some vertex

Matrix : Begin by scanning column1 of matrix A, the only which connects to is vertex 5. Now scan row 1, the only path from to is to vertex 3. Since we have established that a path of length 2 lies between and , we update matrix see Table 2.3.

Table 2.3. Matrix

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A1 | 1 | 2 | 3 | 4 | 5 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 0 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 |
| 5 | 1 | 0 | 1 | 0 | 0 |

Matrix : Next create matrix , using vertex 2 as the pivot point. Begin by scanning column *2* of matrix A, the which connect to are vertices 3 and 4. Now scan *row 2*, only 1 path from exists to = vertex 4. We have now established paths between the following vertices. to and to

Newly added paths have been highlighted in gray. Notice that each new path created is being built upon previously existing paths.

Table 2.4. Matrix

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A2 | 1 | 2 | 3 | 4 | 5 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 1 | 0 |
| 5 | 1 | 0 | 1 | 0 | 0 |

Matrix Matrix will use vertex 3 as the pivot point. As before, scan column 3 to see which vertices connect to . In this case, vertices 1 and 5 have a path to 3. Now, scanning row 3, connects to vertices 2, 4, 5. We have established paths between the following.

Connects to and connects to

Connects to and connects to

Connects to and connects to

Table 2.5. Matrix

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A3 | 1 | 2 | 3 | 4 | 5 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 1 | 0 |
| 5 | 1 | 1 | 1 | 1 | 1 |

Notice now that some paths have exceeded length 2. This is because the newly established paths are not using just 3 as a pivot point, but also the previous pivots points. For instance, vertex 5 has a path to 4 by travelling along the path of is shown.

If you have not guessed by now, we will be creating 2 more adjacency matrices, and Generally speaking, if a graph has *n* vertices, it requires *n* matrices to produce where is the path matrix. To produce and , proceed as before.

Matrices and : For , first scan column 4. At this point, all vertices now have a path to vertex 4. Scanning row 4, we see that 4 has a path only to vertex 2, indicating that all vertices have a path to 2. However, the only vertex which doesn't already have a path to vertex 2 is 2 itself, so we update the matrix accordingly

Table 2.6. Matrix

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A4 | 1 | 2 | 3 | 4 | 5 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 1 | 0 |
| 5 | 1 | 1 | 1 | 1 | 1 |

Completing this process, scan column 5 to see that vertices 1, 3 and 5 all have paths to vertex 5. Scanning row 5 indicates that 5 has a path to all other vertices. Consequently, we add 1's to rows 1, 3 and 5 to reflect that vertices 1, 3 and 5 have paths to all other vertices.

Table 2.7. matrix

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A5 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 0 | 1 | 0 |
| 5 | 1 | 1 | 1 | 1 | 1 |

This completes the path matrix for Figure 2.5.

1

2

3

4

5

Figure 2.5. The shortest path

## 2.4. Connection Problems On Directed Graph Without Cycles

Consider a tuple given by where is rooted directed graph with as set of vertices , as set of arcs and 0 is the source. We assume that the following conditions M.1 and M.2 hold.

**M.1** (Direct connection possibility) For each .

**M.2** (No cycles) For each such that we have

Further, is a non-negative function on the set of arcs. We call such a tuple with the properties M.1 and M.2 a mountain situation because of the following two reasons.

Each mountain problem as described in Chapter 2 leads to a mountain situation, where N corresponds to the set of agents (houses) in the mountain, 0 to the purifier, A to the set of allowed connections determined by the gravity condition.

, (2.4.1)

(where is the height of house *i*) and by reefs etc. Further describes the costof connecting *i* with *j* via a pipe line. M.1 is demanded and M.2 follows from (2.4.1).

On the other hand, given a mountain situation with the properties M.1 and M.2, there exists a height function such that implies One defines ho as follows: for is the length of a longest path from to to 0 if is a path in A, i.e. for every then the length of this path equals

The interesting problems related to such a mountain situation are.

(1) How to find a 0-connecting subtree of , i.e. a subtree connecting each with 0, with minimum cost?

(2) How to allocate the connection costs in such a tree among the agents? In this section we will solve (1) and the next sections deal with (2).

To avoid too many technicalities we assume in the following that does not only satisfy M.1 and M.2, but also M.3:

**M.3** (Genericity condition) For each k and all

We invite the reader to adjust our results for situations where M.3 does not hold. M.3 gives us the possibility to speak of the best connection of Here given a mountain situation (with property M.3), the next theorem shows that there is a unique optimal tree (with minimum costs), connecting all players in N with the root 0. This tree corresponds to the situation where each agent *k*  N connects himself with his best connection point (Moretti, et al, 2002).

**Theorem 2.4.1.** Let be a mountain situation (satisfying, beside M.1 and M.2, also M.3). Let Then

*(i)* is a 0-connecting subtree of .

*(ii)* The tree is the unique 0-connecting subtree with minimum cost.

**Example 2.4.2.** Figure 2.6. corresponds to a mountain situatation where and for each Then the intrinsic height function described by for each . Since the tree with, is an optimal 0-connecting tree with costs

See Figure 2.6.



Figure 2.6. The mountain situation of Example 2.4.2

# 

**3. THE SHAPLEY VALUE AND THE PAYOFF SHARING SCHEME**

**3.1. The Shapley Value**

This value known as the Shapley value after its inventor Lloyd S. Shapley. The Shapley value is a solution concept that is usually formulated with respect to the grand coalition. It defines a way of distributing the value that could be obtained by the grand coalition. For this reason, it is often implicitly assumed that the games to which it is applied are superadditive However, the Shapley value is well-defined for non-superadditive games as well. The Shapley value is based on the witted that the payment that each agent receives should be proportional to his contribution. Application of this idea would be to pay each agent according to how much he increases the value of the coalition of all other players when he joins it, i.e., set the payoff of the player i to . However, under this payoff plan the total payoff assigned to the agents may differ from the value of the grand coalition (Chalkiadakis et al., 2011)‏.

We note that each agent’s payoff under this plan would be 0, while the value of the grand coalition is 1 see Example 3.1.1.

**Example 3.1.1.** Consider a game where and otherwise. The characteristic function of the game as follows.

Tabal 3.1. Coalitions of example 3.1.1

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| S |  |  |  |  |  | ({1,3}) | ({2,3}) | ({N}) |
| *v*(S) | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |

To avoid this problem, we give an ordering of the agents and pay each agent according to how much he contributes to the coalition formed by his previous place in this ordering. Therefore, agent1 receives agent 2 receives etc. However, there is another problem in this situation: two agents that play symmetric roles in the game may receive very different payoffs. If we look at Example 3.1.1. Again under the order-based payoff plan one agent receive 1, whereas two other agents receive 0. In other words, the agents' payoffs in this plan strongly depend on the selected ordering of the agents. Shapley’s vision, which led to the definition of the Shapley value, is that this dependence can be eliminated by averaging over all possible orderings, or permutations for the players. Before defining the Shapley value, we give some notations. Characteristic function of a game is the marginal vector is

is a permuation.

The family of all permutations of of .

: Is the set consists of all precedessors of with respect to permuations

: Is the entrance number of player .

Let the players enter a room one by one in the order and each player give the marginal contribution he creats by entering. Hence , is the vector in with,

.

.

.

.

Let us take then,

**Definition 3.1.2.** The Shapley value of a game is the average of all marginal vectors of the game (Branzei, 2009).

, and the Shapley valueof a player is denoted by, and

. ( Lloyd S. Shapley, 1953)

**Example 3.1.3.** Let be the 3-person game with for each

Let us clculate the Shapley Value.

Since number of players then ther are six permutation to them order

Then, we find the marginal vector for each order as follows:

When

2

Then the marginal vector for is

When

Then the marginal vector for is

When

Then the marginal vector for

When

The marginal vector for is

When

The marginal vector foris .

When

The marginal vector for

We illustrate these results in Table 3.2.

Table 3.2. The marginal vector of the Example 3.1.3.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  | 0 | 4 | 16 |
|  | 0 | 13 | 7 |
|  | 4 | 0 | 16 |
|  | 5 | 0 | 15 |
|  | 7 | 13 | 0 |
|  | 5 | 15 | 0 |
|  | 21 | 45 | 54 |

Shapley value is the average of the marginal vectors

**Example 3.1.4.** Let be the 3-person game with . Now, we calculate the Shapley value of this game.

Since is the number of players then there are six permutation

First, we find the marginal vectors for each order as it is shown below:

When

The marginal vector for is.

When

,

Then the marginal vector for is .

When

,

,

Then the marginal vector for is When

Then the marginal vector for is .

Then the marginal vector for is

When

Then the marginal vector for

We show these results in the following Table 3.3.

Table 3.3. The margiinal vector for Example 3.1.4.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | |  |  | |  |
|  | | 10 | 40 | | 52 |
|  | | 10 | 52 | | 40 |
|  | | 30 | 20 | | 52 |
|  | | 52 | 20 | | 30 |
|  | | 20 | 52 | | 30 |
|  | | 52 | 20 | | 30 |
|  | 174 | | 204 | 234 | |

The Shapley value is the average of the marginal vectors,

## 3.2. The Payoff Sharing Scheme

Now, we find the Shapley value for the set coalition {1,2} . The third player has not entered in this coalition. Therefore we put to him signal ( ) means the player has not entered the coaliton.

The order permutation to set elements of coalition {1,2} are:

When

Then the maginal vector for is (10,40).

When

Then the marginal vector for is

The marginal vectors for the orders can be seen in Table 3.4.

Table 3.4. The marginal vector for coalition {1,2}.

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  | 10 | 40 |
|  | 30 | 20 |
|  | 40 | 60 |

The Shapley value is,

Therefore the Shapley value for the coalition is

We find the Shapley value for the coalition The scond player has not entered in this coalition.

The order permuation to set elements of coalition {1,3} are

When

Then the marginal vector for is (10,40).

When

=20,

Then the marginal vector for is (20,30).

The marginal vectors for the orders can be seenin in Table 3.5.

Table 3.5. The marginal vector for coalition {1,3}.

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  | 10 | 40 |
|  | 20 | 30 |
|  | 30 | 70 |

The Shapley value is

We find the Shapley value for the set coalition {2,3}. The first player has not entered in this coalition.

The order permuation to set elements of coalition {2,3} are

When

Then the marginal vector for is (20,30) .

When

Then the marginal vector for is (20,30) .

The marginal vectors for the orders can be seen The table 3.6.

Table 3.6. The marginal vector for coalition {2,3}.

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  | 20 | 30 |
|  | 20 | 30 |
|  | 40 | 60 |

The Shapley value is,

Let's find the Shapley value for coalitions {1},{2},{3} , in this case the Shapley value is equal to the value of a single coalition, the Shapley value for coalition {1} is , the Shapley value for coalition {2} is the Shapley value for coalition {3} is v(3). At that time rows for the coalition {1},{2},{3} respectively are , and . Based on this information, payoff sharing scheme is given in the Table 3.7.

Table 3.7. The payoff sharing scheme.

|  |  |  |  |
| --- | --- | --- | --- |
| **Coalition** | **Player 1** | **Player 2** | **Player 3** |
| N | 29 | 34 | 39 |
| {1,2} | 20 | 30 | \* |
| {1,3} | 15 | \* | 35 |
| {2,3} | \* | 20 | 30 |
| {1} | 10 | \* | \* |
| {2} | \* | 20 | \* |
| {3} | \* | \* | 30 |

From this study, we see that the Shapley value is an advantage in the distribution of profit or costs in a fair and encouragest way to form the grand coalition. For example the first player alone gets a profit of 10 dollars and when makes a coalition with the second player ({1,2} coalition) gets a profit of 20 dollers, when makes a coalition with the third player ( {1,3} coalition ) gets a profit of 15 dollars and gets a profit of 29 dollars in the grand coalition. Then, the Shapley value is the good solution for this example.

**4. AN APPLICATION OF A MOUNTAIN SITUATION IN ISPARTA**

# 4.1. Introduction

In this chapter we use cooperative game theory and the Shapley value to apply connection problems to the specific areas of the Isparta province on a large stadium in the city center. Here, we achieve conditions of the mountains situation. By linking the main areas in the city of Isparta with shortest path, in order to transfer the population to a safe place in case of emergency. Isparta is Turkish mountains province, located in the southwest of Turkey, an area of it ( 8993) square kilometers, a population of nearly half a million people. This study aims to find a solution that would satisfy residents of these areas to take them to a safe place in case of emergency, such as earthquakes and tremors because of sometimes this city exposed to earthquakes causing fear and panic the population and all of pepole left them houses to search for safe place. This problem has stimulated the researcher to study it in game theory field to find optimal solution connects residential areas with safe area that contain amenitties , where he has been studing the subject in collaboration with us engineering division in the municipality of the province, and provided us with maps and information about the link. The formulation of the problem according to the following example.

## 4.2. Data and Solution

Suppose that we divide Isparta city into five basic areas according to the road map. They are as follows and as shown in Figure 4.1and4.2.

**Region 1:** Akyol cemetery (Ayazma neighborhood, Vatan Neighborhood, Halikent Neighborhood, and Gülcü Neighborhood).

**Region 2:** Asriy Cemetery (Yedişehitler Neighborhood, Muzaffer Neighborhood, Işıkkent Neighborhood, Batıkent Neighborhood Binbirevler Neighborhood, Hızır Neighborhood, Bağlar Neighborhood).

**Region 3:** Otogar (Fatih Neighborhood, Anadolu Neighborhood,Bahçelievler Neighborhood, Modernevler Neighborhood, Zafer Neighborhood).

**Region 4:** Çünür Cemetery (Mehmet Tönge Neighborhood, Çünür Neighborhood Kayı Village).

**Source 0 :** Stadyum (Çelebi Neighborhood, Turan Neighborhood, Neighborhood, Davraz Neighborhood, Pirimehmet Neighborhood, Yayla Neighborhood).

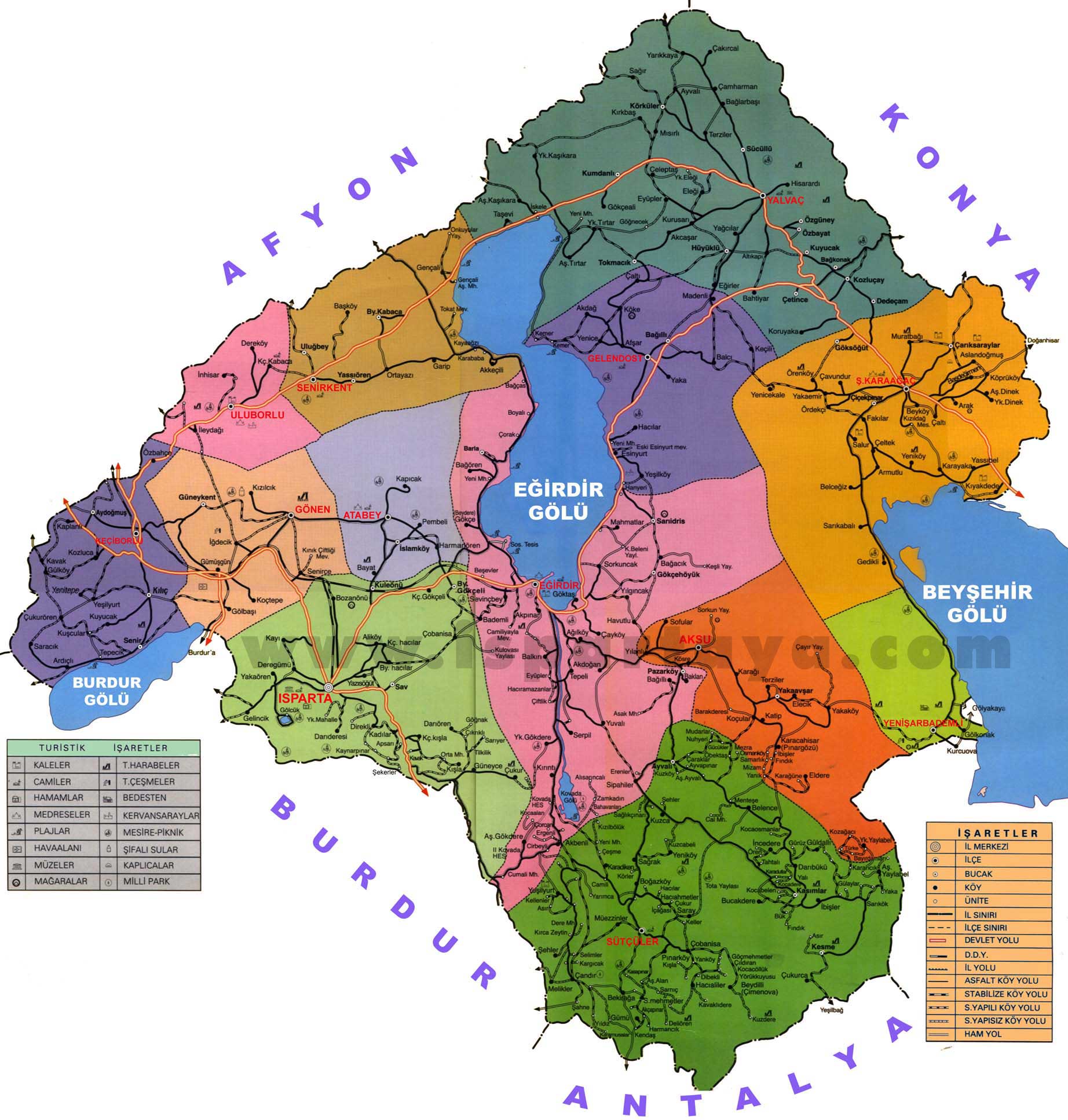
See the map of Isparta city in the Figure 4.1.

Figure 4.1. Map of Isparta city

We calculate the Shapley value to solve this problem. The directed weighted graph in Figure 10.

(The vertices set {1,2,3,4}) are the main regions, where the people can meet. This means N={1,2,3,4}.

(The vertix {0}) is the source (Atatürk Stadium)).

(The edges set) of these points {0,1,2,3,4} can be written as

A={(1.0), (2.0), (2,1), (3,0), (3,1), (3,2), (4,0), (4,1), (4,2), (4,3)}.

(The numbers on the edges) indicate to the distances between any two vertices, this means the characteristic function is ,.

Then, is a mountain situation because of the satisfied condition **M.1,M.2,M.3** of a mountain situation. (see Figure 9 and 10).

Since ( direct connection possiblity) then **M.1.** condition is nsatisfied

Since no cycles between the player, then **M.2.** condition is satisfied. Also **M.3.** is satisfied.

Since, the above conditions hold, is mountain situaton. We can modeling this problem by using cooperative game theory.

Let us consider a mountaion situation Here, is the set of players and the coalitional values are given as follows: Now, we compute the Shapley value of this problem.



Figure 4.2. Main areas of Isparta scheme

We have four types of coalitions:

Single coalitions are {1},{2},{3},{4}. For example  *v* ({1})=3.4 which can be seen in Figure 4.3.

3.4

Figure 4.3. Single coalition

Bi coalitions are {1,2},{1,3},{1,4},{2,3},{2,4},{3,4}. For example *v* ({1,3}) = 8.2 which can be seen in Figure 4.4.

4.8

3.4

Figure 4.4. Bi coalition

Tri coalitions are {1,2,3},{1,2,4},{1,3,4},{2,3,4}. For example *v* ({2,3,4})=10.7 which can be seen in Figure 4.5.

4.7

3

3

Figure 4.5. Tri coalition

Quadrilateral coalition ( N- coalition) for players {1,2,3,4}. For example *v* ({1,2,3,4})=16.1 which can be seen in Figure 4.6.

4.7

Figure 4.6. N-coalition

3.4

5

3

When

There are twenty-four permutations. These permutations are as follows:

The marginal vector is

When

,

Then the marginal vector is (3.4,5,0.7).

When

,

Then the marginal vector is (3.4,3.2,4.8.4.7 ).

When

Then the marginal vector is (3.4,3.2,4.8,4.7)

When

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Then the marginal vector is (3.4,2.8,0.7,9.2)

When

Then the marginal vector is (3.4,3.2,0.3,9.2).

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Then the marginal vector is

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We illustrate the marginal vector of all permutations in the Table 4.1.

Table 4.1. The marginal vector for game

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  | 3.4 | 5 | 3 | 4.7 |
|  | 3.4 | 5 | 0.7 | 7 |
|  | 3.4 | 3.2 | 4.8 | 4.7 |
|  | 3.4 | 3.2 | 4.8 | 4.7 |
|  | 3.4 | 2.8 | 0.7 | 9.2 |
|  | 3.4 | 3.2 | 0.3 | 9.2 |
|  | 5.4 | 3 | 3 | 4.7 |
|  | 5.4 | 3 | 0.7 | 7 |
|  | 5.4 | 3 | 3 | 4.7 |
|  | 5.4 | 3 | 3 | 4.7 |
|  | 5.4 | 3 | 0.7 | 7 |
|  | 5.4 | 3 | 0.7 | 7 |
|  | 5.7 | 3.2 | 2.5 | 4.7 |
|  | 5.7 | 3.2 | 2.5 | 4.7 |
|  | 5.4 | 3.5 | 2.5 | 4.7 |
|  | 5.4 | 3.5 | 2.5 | 4.7 |
|  | 5.7 | 3.2 | 2.5 | 4.7 |
|  | 5.4 | 3.5 | 2.5 | 4.7 |
|  | 5.6 | 2.8 | 0.7 | 7 |
|  | 5.6 | 3.2 | 0.3 | 7 |
|  | 5.4 | 3 | 0.7 | 7 |
|  | 5.4 | 3 | 0.7 | 7 |
|  | 5.7 | 3.2 | 0.2 | 7 |
|  | 5.4 | 3.5 | 0.2 | 7 |
|  | 119.2 | 79.2 | 43.2 | 144.8 |

The Shapley value is:

# 5. CONCLUSION AND FUTURE WORK

In this thesis, we have studied the concepts of the main solutions for cooperative game theory. The concept behind all these are some of the rationality presumed players which is as follows:

It was reached that the class of cooperative games from the game theory,by using the Shapley value to treatment the problem of cost-sharing or profit in fair way satisfying all players.

After presenting some basic concepts of cooperative game, and discussing applications to solve them we have found results in the case of a coalition among players which better than individual cases.

From the figure corresponds to a mountain situation situatation where N={1,2,3}, A={(1,0),(2,0),(2,1),(3,0),(3,1),(3,2)}, the payoff vector (10,15,20) is the optimal 0-connecting tree with cost 10 + 15+ 20 = 45. When we calculate the Shapley value of the same problem

,

and (the cost).

This thesis offers many open problems. We believe that the cooperative interval theory is a promising area of applied mathematics with many open problems and high potential for application in the real world problems. After this thesis we plan to attack some of the open problems in the near future.

# 

# REFERENCES

Alparslan Gök, S.Z., Palancı, O., Olgun, M.O., 2014b. Cooperative interval game: mountain situations with interval data. Journal of Computational and applied Mathematics, 259, part B: 622-632.

Alparslan Gök, S.Z., Miquel, S. and Tijs, S., 2009d. Cooperation under interval uncertainty. Mathematical Methods of Operations Research, 69, 99-109..

Bird, C.G., 1976. On cost allocation for a spanning tree: a game theoretic ap-proach. Networks, 6, 335-350.

Bok, J., & Hladík, M., 2015. Selection-based approach to cooperative interval games. In Operations Research and Enterprise Systems (pp. 40-53). Springer International Publishing.‏

Bondareva, O. N., 1963. Some applications of linear programming methods to the theory of cooperative games. Problemy kibernetiki 10 , 119-139

Branzei, R., Dimitrov, D., & Tijs, S., 2003. Shapley-like values for interval bankruptcy games. Economics Bulletin, 3(9), 1-8.

Branzei, R., Dimitrov, D., & Tijs, S., 2008. Models in cooperative game theory (Vol. 556). Springer Science & Business Media.‏

Chalkiadakis, G., Elkind, E., & Wooldridge, M., 2011. Computational aspects of cooperative game theory. Synthesis Lectures on Artificial Intelligence and Machine Learning, 5(6), 1-168.

Curiel, I., 2013. Cooperative game theory and applications: cooperative games arising from combinatorial optimization problems (Vol. 16). Springer Science & Business Media.‏

Debreu, G., & Scarf, H.,1963. A limit theorem on the core of an economy. International Economic Review, 4(3), 235-246.

Dubey, P., 1975. On the uniqueness of the Shapley value. International Journal of Game Theory 4.3, 131-139.

Fragnelli, V., & Moretti, S., 2008. A game theoretical approach to the classification problem in gene expression data analysis. Computers & Mathematics with Applications, 55(5), 950-959.‏

Gansner, E. R., Koutsofios, E., North, S. C., & Vo, G. P., 1993. A technique for drawing directed graphs. Software Engineering, IEEE Transactions on, 19(3), 214-230.

Gillies, D.B., 1959. Solutions to general non-zero-sum games. In Kuhn, H.W., Tucker A.W. (Ed.), Contributions to Theory of Games Vol IV. (47-85), Princeton University Press, 468p, Princeton.

Harsanyi J.C., 1959. A bargaining model for the cooperative n-person game. In Kuhn, H.W., Tucker A.W. (Ed.), Contributions to Theory of Games Vo IV. (325-355), Princeton University Press, 468p, Princeton.

Lu, H., Mikkilineni, K., & Richardson, J. P., 1987, February. Design and evaluation

of algorithms to compute the transitive closure of a database relation. In Data Engineering, 1987 IEEE Third International Conference on (pp. 112-119). IEEE

Moretti, S., Norde, H., Do, K.H.P., Tijs, S., 2002. Connection problems in mountains and monotonic allocation schemes. TOP, 10(1), 83-99.

Moretti, S., et al., 2004. The P-value for cost sharing in minimum cost spanning tree situations. Springer US.

Moretti, S., 2009 Game Theory applied to gene expression analysis. 4OR 7.2.

Moretti, S., Alparslan Gök, S. Z., Branzei, R., Tijs, S., (2011). Connection situations

under uncertainty and cost monotonic solutions. Computers and Operations Research, 38(11), 1638-1645.

Neyman A., 1985 Jan. Bounded complexity justifies cooperation in the finitely repeated prisoners' dilemma. Economics letters. 1;19(3):227-9.

Nikolidaki, Aikaterini, and Dimitris Fotakis., 2015. Game Theoretic Models for Power Control in Wireless Networks.

Norde, H., Moretti, S., Tijs, S., 2001. Minimum cost spanning tree games and population monotonic allocation schemes. CentER Discussion Paper, 18, Tilburg University, Tilburg, The Netherlands.

Palancı, O., Alparslan Gök, S. Z., Weber, G.-W., 2013. Forest situations and cost monotonic solutions. In Petrosyan, L.A., Zenkevich, N. A. (Ed.), Contributions to game Theory and Management Vol. VI (351-361), Graduate.

Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal of Applied Mathematics, 17(6), 1163-1170.

Shapley, L. S., 1953. Stochastic games. Proceedings of the National Academy of Sciences, 39(10), 1095-1100.

Shapley, L. S., 1953. Contributions to the Theory of Games II. Value for n-person

games. Annals of mathematics studies 28: 307-317.

Shapley, L. S., 1967. On balanced sets and cores. Naval research logistics quarterly, 14(4), 453-460.‏

Suijs, J., 2003. Cost allocation in spanning network enterprises with stochastic connection costs. Games and Economic Behavior, 42(1), 156-171.

von Neumann, J., Morgenstern, O., 1944. Theory of Games and Economic Behavior. Princeton University Press, 776p, Princeton.

Tijs, S., Branzei, R., Moretti, S., Norde, H., 2006. Obligation rules for minimum cost spanning tree situations and their monotonicity properties. European Journal of Operational Research, 175(1), 121-34.

Umeyama, S., 1988. An eigendecomposition approach to weighted graph matching problems. Pattern Analysis and Machine Intelligence, IEEE Transactions on, 10(5), 695-703

((<http://www.math.uri.edu/~kulenm/mth381pr/GAMETH/gametheory.html)>)

(([www.rijpm.com/.../Open\_Options\_Introduction\_To\_Game\_Theor)](http://www.rijpm.com/.../Open_Options_Introduction_To_Game_Theor)))*.*

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